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On Weakly Hurewicz Spaces

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Abstract. A space X is *weakly Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are a dense subset $Y \subseteq X$ and finite subfamilies $\mathcal{V}_n \subseteq \mathcal{U}_n (n \in \mathbb{N})$ such that for every point of Y is contained in $\bigcup \mathcal{V}_n$ for all but finitely many n. In this paper, we investigate the relationship between Hurewicz spaces and weakly Hurewicz spaces, and also study topological properties of weakly Hurewicz spaces.

1. Introduction

By a space, we mean a topological space. In 1925, Hurewicz [2] (see also [3, 5]) defined a space X to be *Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n. Clearly every Hurewicz space is Lindelöf. As a generalization of Hurewicz spaces, the authors [7] defined a space X to be *almost Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup\{\overline{V} : V \in \mathcal{V}_n\}$ for all but finitely many n. Kočinac [4] defined (see also [6]) a space X to be *weakly Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are a dense subset $Y \subseteq X$ and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $y \in \bigcup \mathcal{V}_n$ for all but finitely many n. From the definitions above, we can easily see that the Hurewicz property implies the almost Hurewicz property and the weak Hurewicz property. The authors [7] showed that every regular almost Hurewicz space is Hurewicz and gave an example that there exists a Urysohn almost Hurewicz space that is not Hurewicz. On the study of Hurewicz and almost Hurewicz spaces, the readers can see the references [2, 3, 5, 6, 7]. The purpose of this paper is to investigate the relationship between Hurewicz spaces and weakly Hurewicz spaces, and also study topological properties of weakly Hurewicz spaces.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [1].

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2. Properties of Weakly Hurewicz Spaces

In this section, first we give an example showing that Tychonoff weakly Hurewicz spaces need not be Hurewicz.

Lemma 2.1. *If a space X has a σ-compact dense subset, then X is weakly Hurewicz.*

Proof. Let $D = \bigcup_{n\in\mathbb{N}} D_n$ be a σ-compact dense subset of X, where each D_n is a compact subset of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let us consider the dense subset D of X. For each $n \in \mathbb{N}$, the set $\bigcup_{m\leq n} D_m$ is compact, there exists a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\bigcup_{m\leq n} D_m \subseteq \bigcup \mathcal{V}_n$. Thus we get a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in D$, there exists an $n_0 \in \mathbb{N}$ such that $x \in D_{n_0}$, thus $x \in \bigcup \mathcal{V}_n$ for all $n \geq n_0$, which shows that X is weakly Hurewicz. □

Since every separable space has a countable dense subset, thus we have the following Corollary by Lemma 2.1

Corollary 2.2. *Let X be a separable space. Then X is weakly Hurewicz.*

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.3. *There exists a Tychonoff weakly Hurewicz space X that is not Hurewicz.*

Proof. Let D be the discrete space of cardinality ω_1, let

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product of βD and $\omega + 1$. Then X is weakly Hurewicz by Lemma 2.1, since $\beta D \times \omega$ is a σ-compact dense subset of X. Since $D \times \{\omega\}$ is an uncountable discrete closed subset of X, X is not Lindelöf. Thus X is not Hurewicz, since every Hurewicz space is Lindelöf, which completes the proof. □

From Example 2.3, it is not difficult to see that the closed subset of a Tychonoff weakly Hurewicz space need not be weakly Hurewicz, since $D \times \{\omega\}$ is an uncountable discrete closed subset of X. In the following, we give a positive result. We omit the proof of the following lemma which can be easily proved. A subset B of a space X is *regular closed* if $B = \overline{B^o}$.

Lemma 2.4. *Let F be a regular closed subset of a space X and Y a dense subset of X. Then $Y \cap F$ is a dense subset of F.*

Theorem 2.5. *Every regular closed subset of a weakly Hurewicz space is weakly Hurewicz.*

Proof. Let X be a weakly Hurewicz space and F be a regular closed subset of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of F. For each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$, there exists an open subset $V_{(n,U)}$ of X such that $V_{(n,U)} \cap F = U$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \{V_{(n,U)} : U \in \mathcal{U}_n\} \cup \{X \setminus F\}$, $\mathcal{U}'_n$ is an open cover of X. Then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of open covers of X. There exist a dense subset Y of X and a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and for every point of Y is contained in $\bigcup \mathcal{V}'_n$ for all but finitely many n, since X is weakly Hurewicz. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \mathcal{V}'_n \setminus \{X \setminus F\}$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{W \cap F : W \in \mathcal{W}_n\}$. Then $\mathcal{V}_n$ is a finite subset $\mathcal{U}_n$. By Lemma 2.4, the set $Y \cap F$ is a dense subset of F. For each $y \in Y \cap F$ and for each $n \in \mathbb{N}$, if $y \in \bigcup \mathcal{V}'_n$, then $y \in \bigcup \mathcal{W}_n$, hence $y \in \bigcup \mathcal{V}_n$. This shows that for $y \in Y \cap F$, $y \in \bigcup \mathcal{V}_n$ for all but finitely many n, which completes the proof. □

Corollary 2.6. *If X is a weakly Hurewicz space, then every open and closed subset of X is weakly Hurewicz.*

Theorem 2.7. *Every Tychonoff space X can be represented as a closed G_δ subspace in a Tychonoff weakly Hurewicz space.*

Proof. Let X be a Tychonoff space and let

$$R(X) = (\beta X \times (\omega + 1)) \setminus ((\beta X \setminus X) \times \{\omega\})$$

be the subspace of the product of βX and $\omega + 1$.

Thus X can be represented as a closed G_δ subspace in $R(X)$ by the definition of the topology $R(X)$, since X is homeomorphic to $X \times \{\omega\}$ and $X \times \{\omega\}$ is a closed G_δ subset of $R(X)$.

Similarly to the proof that X in Example 2.3 is weakly Hurewicz, it is not difficult to show that $R(X)$ is weakly Hurewicz. □

Since a continuous image of a Hurewicz space is Hurewicz, similarly we have the following result.

Theorem 2.8. *A continuous image of a weakly Hurewicz space is weakly Hurewicz.*

Proof. Let X be a weakly Hurewicz space, $f : X \to Y$ be continuous and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of open covers of X. There are a dense subset D of X and a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in D$, $x \in \bigcup\{f^{-1}(U) : U \in \mathcal{V}_n\}$ for all but finitely many n, since X is weakly Hurewicz. Thus the dense subset $f(D)$ of Y and the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witness for $(\mathcal{U}_n : n \in \mathbb{N})$ that Y is weakly Hurewicz. Indeed, let $y \in f(D)$, there exists $x \in D$ such that $f(x) = y$. Then $x \in \bigcup\{f^{-1}(U) : U \in \mathcal{V}_n\}$ for all but finitely n. Thus $y = f(x) \in \cup\{U : U \in \mathcal{V}_n\}$. This shows that $y \in \bigcup\{U : U \in \mathcal{V}_n\}$ for all but finitely n, which completes the proof. □

Next we turn to consider preimages. To show that the preimage of a weakly Hurewicz space under a closed 2-to-1 continuous map need not be weakly Hurewicz. We use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the form $(U \times \{0\}) \bigcup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X.

Example 2.9. *There exists a closed 2-to-1 continuous map $f : A(X) \to X$ such that X is a Tychonoff weakly Hurewicz space, but $A(X)$ is not weakly Hurewicz.*

Proof. Let X be the space X of Example 2.3. Then X is weakly Hurewicz and has an infinite discrete closed subset $A = \{\langle d_\alpha, \omega\rangle : \alpha < \omega_1\}$. Hence $A \times \{1\}$ is not weakly Hurewicz, since $A \times \{1\}$ is an uncountable infinite discrete, open and closed subset in $A(X)$. Thus the Alexandroff duplicate $A(X)$ of X is not weakly Hurewicz by Corollary 2.6. Let $f : A(X) \to X$ be the projection. Then f is a closed 2-to-1 continuous map, which completes the proof. □

In the following, we give a positive result.

Theorem 2.10. *If $f : X \to Y$ is an open and perfect continuous mapping and Y is a weakly Hurewicz space, then X is weakly Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Then for each $y \in Y$ and each $n \in \mathbb{N}$, there is a finite subfamily $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that

$$f^{-1}(y) \subseteq \bigcup \mathcal{U}_{n_y}.$$

Let $U_{n_y} = \bigcup \mathcal{U}_{n_y}$. Then $V_{n_y} = Y \setminus f(X \setminus U_{n_y})$ is an open neighborhood of y, since f is closed.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$. Then $\mathcal{V}_n$ is an open cover of Y. Thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of Y. There are a dense subset D of Y and a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in D$, $y \in \bigcup\{V : V \in \mathcal{V}'_n\}$ for all but finitely many n, since Y is weakly Hurewicz. Without loss of generality, we may assume that $\mathcal{V}'_n = \{V_{n_{y_i}} : i \leq n'\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \bigcup_{i \leq n'} \mathcal{U}_{n_{y_i}}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Since f is an open mapping, thus $f^{-1}(D)$ is a dense subset of X. Therefore the subset $f^{-1}(D)$ and the sequence $(\mathcal{U}'_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$

that X is weakly Hurewicz. Indeed, let $x \in f^{-1}(D)$, $f(x) \in D$. Then $f(x) \in \bigcup\{V_{n_{y_i}} : i \leq n'\}$ for all but finitely many n. For $n \in \mathbb{N}$, if $f(x) \in \bigcup\{V_{n_{y_i}} : i \leq n'\}$, then there exists some $i \leq n'$ such that $f(x) \in V_{n_{y_i}}$. Hence

$$x \in f^{-1}(f(x)) \in f^{-1}(V_{n_{y_i}}) \subseteq U_{n_{y_i}} \subseteq \bigcup \mathcal{U}_{n_{y_i}}.$$

Therefore $x \in \bigcup\{U : U \in \mathcal{U}'_n\}$. This shows that $x \in \bigcup\{U : U \in \mathcal{U}'_n\}$ for all but finitely n, which completes the proof. □

It is well-known that the product of a Hurewicz space and a compact space is Hurewicz. For weakly Hurewicz spaces, we have the similar result by Theorem 2.10.

Corollary 2.11. *If X is a weakly Hurewicz space and Y is a compact space, then $X \times Y$ is weakly Hurewicz.*

Lemma 2.12. *The weakly Hurewicz property is closed under countable union.*

Proof. Let $X = \bigcup\{X_k : k \in \mathbb{N}\}$, where each X_k is weakly Hurewicz. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of X. For each $k \in \mathbb{N}$, let us consider the sequence $\{\mathcal{U}_n : n \geq k\}$. For each $k \in \mathbb{N}$, since X_k is weakly Hurewicz, there are a dense subset D_k of X_k and a sequence $(\mathcal{V}_{n,k} : n \geq k)$ such that for each $n \geq k$, $\mathcal{V}_{n,k}$ is a finite subset of $\mathcal{U}_n$ and for each $x \in D_k$, $x \in \bigcup \mathcal{V}_{n,k}$ for all but finitely many $n \geq k$. Let $D = \bigcup_{k \in \mathbb{N}} D_k$. Then D is a dense subset of X. For each $n \in \mathbb{N}$, let $\bigcup\{\mathcal{V}_{n,j} : j \leq n\}$. Then each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. The dense subset D of X and the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witness that X is weakly Hurewicz. If fact, for each $x \in D$, there exists some $k \in \mathbb{N}$ such that $x \in D_k$, thus $x \in \bigcup \mathcal{V}_n$ for all but finitely many $n > k$. which completes the proof. □

We have the following result by Lemma 2.12 and Corollary 2.11.

Corollary 2.13. *If X is a weakly Hurewicz space and Y is a σ-compact space, then $X \times Y$ is weakly Hurewicz.*

Theorem 2.14. *The following are equivalent for a space X.*
(*a*) *X is Hurewicz;*
(*b*) *$A(X)$ is Hurewicz;*
(*c*) *$A(X)$ is weakly Hurewicz.*

Proof. (*a*) → (*b*). We show that $A(X)$ is Hurewicz. To this end, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $A(X)$. For each $n \in \mathbb{N}$ and each $x \in X$, choose an open neighborhood $W_{n_x} = (V_{n_x} \times \{0, 1\}) \setminus \{\langle x, 1\rangle\}$ of $\langle x, 0\rangle$ satisfying that there exists some $U_{n_x} \in \mathcal{U}_n$ such that $W_{n_x} \subseteq U_{n_x}$, where V_{n_x} is an open neighborhood of x in X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$. Then $\mathcal{V}_n$ is an open cover of X. Thus $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of X, there exists a sequence $(E_n : n \in \mathbb{N})$ of finite subsets of X such that $(\{V_{n_x} : x \in E_n\})$ witnesses the Hurewicz property of X. For each $n \in \mathbb{N}$ and each $x \in E_1 \cup E_2 \cup \cdots \cup E_n$, pick $U'_{n_x} \in \mathcal{U}_n$ such that $\langle x, 1\rangle \in U'_{n_x}$. For each $n \in \mathbb{N}$, let

$$\mathcal{U}'_n = \{U_{n_x} : x \in E_n\} \bigcup \{U'_{n_x} : x \in E_1 \cup E_2 \cup \cdots \cup E_n\}.$$

Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Thus the sequence $(\mathcal{U}'_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that $A(X)$ is Hurewicz.

(*b*) → (*c*). It is trivial.

(*c*) → (*a*). For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U^n_x : x \in X\}$ be an open cover of X, where U^n_x is an open neighborhood of x. Applying the weak Hurewicz property of $A(X)$ to the open covers of $A(X)$ below

$$\{(U^n_x \times \{0, 1\}) \setminus \{\langle x, 1\rangle\} : x \in X\} \cup \{\{\langle x, 1\rangle\} : x \in X\},$$

we have a dense subset D in $A(X)$ and a sequence $(F_n : n \in \mathbb{N})$ of finite sets in X such that D and $\{(U^n_x \times \{0, 1\}) \setminus \{\langle x, 1\rangle\} : x \in F_n\} \cup \{\{\langle x, 1\rangle\} : x \in F_n\}$ witness the weak Hurewicz property of $A(X)$ for the given cover above. Let $\mathcal{V}_n = \{U^n_x : x \in F_n\}$. For each $n \in \mathbb{N}$, take a finite subfamily $\mathcal{W}_n \subseteq \mathcal{U}_n$ satisfying $F_1 \cup F_2 \cup \cdots \cup F_n \subseteq \bigcup \mathcal{W}_n$. If $x \in \bigcup_{n \in \mathbb{N}} F_n$, then $x \in \bigcup \mathcal{W}_n$ for all but finitely many $n \in \mathbb{N}$. If $x \in X \setminus \bigcup_{n \in \mathbb{N}} F_n$, then by $\langle x, 1\rangle \in D$ we have $x \in \bigcup \mathcal{V}_n$ for all but finitely many $n \in \mathbb{N}$. □

Remark 2.15. *However the Alexandorff duplicate $A(X)$ of a Tychonoff weakly Hurewicz space X need not be weakly Hurewicz. For example, let P be the space of irrationals, then it is a Lindelöf weakly Hurewicz space such that $A(P)$ is not weakly Hurewicz.*

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